Chapter 3

Limit and Continuity

Contents

The development of calculus was stimulated by two geometric problems: finding areas of plane regions and finding tangent lines to curves. Both of those problems require a limit process for their general solution. However, limit process occurs in many other applications as well. Besides the concept of limit is the fundamental building block on which all other calculus concepts are based.

3.1 Definition of Limit

Limits described what happens to a function $f(x)$ as its variable x approaches to a particular number **a** but not on **a**.

Definition 3.1.1 Let f be a function defined in an open interval containing **a**, with the possible exception of **a** it self. Then, the limit of the function at **a** is the number L, written as

$$\lim_{x\to\mathbf{a}} f(x) = L$$

■ **Example 3.1** Let $f(x) = x+4$, what happens to $f(x)$ as x approaches to 1, but not equal to one.
Solution: To investigate the behavior of $f(x)$ as x approaches 1 numerically and graphically we can construct a table and draw a graph of $f(x)$ for x near 1. The above table is as x approaches 1 from the right The above table is as x approaches 1 from the left.
From the above table, we can conclude that $f(x)$ approaches 5 as x approaches 1 from both the left and right side of 1. ■

x^+	1.5	1.3	1.1	1.01	1.01
f(x)	5.5	5.3	5.1	5.01	5.01

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x^-	0.5	0.7	0.9	0.99	0.999
f(x)	4.5	4.7	4.9	4.99	4.999

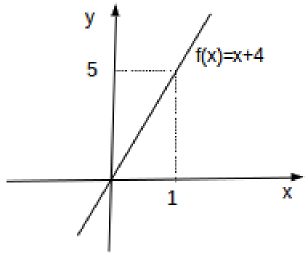

Figure 3.1: $f(x) = x + 4$

■ **Example 3.2** Find $\lim_{x\to 2} f(x)$ for a function $f(x) = x^2$.
Solution: The following table illustrates the behavior of the function, as x becomes closer and closer to 2 from both the left and right side of 2. The above table is as x approaches 2 from the right; that is, $\lim_{x\to 2^+} f(x)$ The above table is as x approaches 2 from the left; that is, $\lim_{x\to 2^-} f(x)$.
Therefore, the value of $f(x) = x^2$ are near 4 whenever x is close to 2 from both the left and right sides

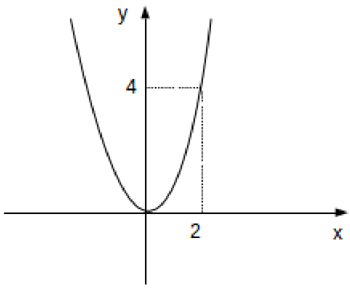

Figure 3.2: $f(x) = x^2$

since $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = \lim_{x\to 2} f(x) = 4$. ■

Definition 3.1.2 Let f be a function defined in an open interval containing **a**, with the possible exception of **a** it self. Then, the limit of the function at **a** is the number L, written as for all number $\varepsilon > 0$, there exist a number $\delta > 0$ such that

$$0 < |x - \mathbf{a}| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

■ **Example 3.3** Evaluate $\lim_{x\to \mathbf{a}} x$.
Solution: Let $\varepsilon > 0$ be any number. Take $\delta = \varepsilon$.
Now

$$\begin{aligned} |x - \mathbf{a}| < \delta &\Rightarrow |x - a| < \varepsilon \\ &\Rightarrow |f(x) - \mathbf{a}| < \varepsilon \text{ since } f(x) = x \end{aligned}$$

Hence, $\lim_{x\to \mathbf{a}} x = \mathbf{a}$. ■

x^+	2.5	2.3	2.1	2.01	2.001	2.0001
f(x)	6.25	5.29	4.41	4.0401	4.004001	4.00040001

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x^-	1.5	1.7	1.9	1.99	1.999	1.9999
f(x)	2.25	2.89	3.61	3.9601	3.9960001	3.99960001

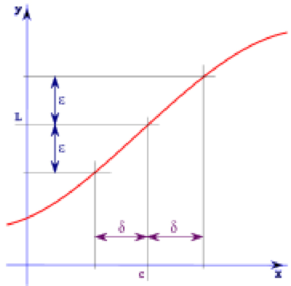

Figure 3.3: The limit of f on the open interval containing **a**

■ **Example 3.4** Show that $\lim_{x\to 5} f(x) = 4$ by using $\varepsilon - \delta$ definition, where $f(x) = 2x - 6$.
Proof: Let $\varepsilon > 0$ be given. We must find $\delta > 0$ such that

$$|x-5| < \delta \Rightarrow |f(x)-4| < \varepsilon$$

Consider

$$\begin{aligned} |f(x)-4| < \varepsilon &\Leftrightarrow |(2x-6)-4| < \varepsilon \\ &\Leftrightarrow |2x-10| < \varepsilon \\ &\Leftrightarrow |x-5| < \frac{\varepsilon}{2} \end{aligned}$$

Now choose $\delta = \frac{\varepsilon}{2}$. Thus,

$$\begin{aligned} 0 < |x-5| < \delta &\Rightarrow |x-5| < \frac{\varepsilon}{2} \\ &\Rightarrow 2|x-5| < \varepsilon \\ &\Rightarrow |2x-10| < \varepsilon \\ &\Rightarrow |(2x-6)-4| < \varepsilon \end{aligned}$$

Therefore, $|x-5| < \delta \Rightarrow |f(x)-4| < \varepsilon$. ■

■ **Example 3.5** Given $\lim_{x\to -2} f(x) = -4$, find $\delta > 0$ for a number $\varepsilon = 0.005$ such that if $0 < |x - \mathbf{a}| < \delta$, then $|f(x) - L| < \varepsilon$.
Solution: For every $\varepsilon = 0.005 > 0$ such that

$$0 < |x-(-2)| < \delta \Rightarrow |f(x)-(-4)| < \varepsilon$$

So that,

$$\begin{aligned} 0 < |x-(-2)| < \delta &\Rightarrow |(5x+6)+4| < 0.005 \\ &\Rightarrow 5|(x+2)| < 0.005 \\ &\Rightarrow |x+2| < \frac{0.005}{5} = 0.001 \end{aligned}$$

Thus, we can choose $\delta = 0.001$ or any positive number less than 0.001. ■

■ **Example 3.6** Show that $\lim_{x\to 4} f(x) = 10$ by using $\varepsilon - \delta$ definition, where $f(x) = x^2 - 6$.
Proof: Let $\varepsilon > 0$ be given. We must find $\delta > 0$ such that

$$|x-4| < \delta \Rightarrow |f(x)-10| < \varepsilon$$

Consider

$$\begin{aligned} |f(x)-10|<\varepsilon &\Leftrightarrow |(x^2-6)-10|<\varepsilon \\ &\Leftrightarrow |x^2-16|<\varepsilon \\ &\Leftrightarrow |(x+4)(x-4)|<\varepsilon \end{aligned}$$

Let choose $\delta_1 = 1$. Thus,

$$\begin{aligned} 0<|x-4|<1 &\Rightarrow -1<x-4<1 \\ &\Rightarrow 3<x<5 \\ &\Rightarrow 7<x+4<9 \\ &\Rightarrow |x+4|<9 \\ &\Rightarrow |x+4||x-4|<9|x-4| \end{aligned}$$

Let for some values of x we have $9|x-4|<\varepsilon \Rightarrow |x-4|<\frac{\varepsilon}{9}$. So choose $\delta_2=\frac{\varepsilon}{9}$.
Now choose $\delta = \min\{\delta_1, \delta_2\} = \{1, \frac{\varepsilon}{9}\}$.

$$\begin{aligned} 0<|x-4|<\delta=\frac{\varepsilon}{9} &\Rightarrow |x-4|<\frac{\varepsilon}{9} \\ &\Rightarrow 9|x-4|<\varepsilon \\ &\Rightarrow |x+4||x-4|<\varepsilon \text{ since } |x-4|<9 \\ &\Rightarrow |(x+4)(x-4)|<\varepsilon \\ &\Rightarrow |x^2-16|<\varepsilon \\ &\Rightarrow |(x^2-6)-10|<\varepsilon \\ &\Rightarrow |f(x)-10|<\varepsilon \end{aligned}$$

Therefore, $|x-4|<\delta \Rightarrow |f(x)-10|<\varepsilon$. ■

■ **Example 3.7** Show that $\lim_{x\to 4} f(x) = \frac{2}{3}$ by using $\varepsilon-\delta$ definition, where $f(x)=\frac{4}{x+2}$.
Proof: Let $\varepsilon>0$ be given. We must find $\delta>0$ such that

$$|x-4|<\delta \Rightarrow \left|f(x)-\frac{2}{3}\right|<\varepsilon$$

Consider

$$\begin{aligned} \left|f(x)-\frac{2}{3}\right|<\varepsilon &\Leftrightarrow \left|\frac{12-2(x+2)}{3(x+2)}\right|<\varepsilon \\ &\Leftrightarrow \left|\frac{8-2x}{3(x+2)}\right|<\varepsilon \\ &\Leftrightarrow \frac{2}{3}\left|\frac{4-x}{x+2}\right|<\varepsilon \\ &\Leftrightarrow \frac{2}{3}\left|\frac{x-4}{x+2}\right|<\varepsilon \end{aligned}$$

Let choose $\delta_1 = 1$. Then,

$$\begin{aligned} |x-4|<\delta_1=1 &\Rightarrow -1<x-4<1 \\ &\Rightarrow 3<x<5 \\ &\Rightarrow 5<x+2<7 \\ &\Rightarrow 5<|x+2|<7 \\ &\Rightarrow \frac{1}{7}<\frac{1}{|x+2|}<\frac{1}{5} \\ &\Rightarrow \frac{1}{|x+2|}<\frac{1}{5} \\ &\Rightarrow \frac{2|x-4|}{3}\frac{1}{|x+2|}<\frac{2|x-4|}{15} \end{aligned}$$

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Let for some values of x we have $\frac{2|x-4|}{15} < \varepsilon \Rightarrow |x-4| < \frac{15}{2}\varepsilon$. So choose $\delta_2 = \frac{15}{2}\varepsilon$.
Now choose $\delta = \min\{\delta_1, \delta_2\} = \{1, \frac{15}{2}\varepsilon\}$.

$$\begin{aligned}
0 < |x-4| < \delta = \frac{15}{2}\varepsilon \quad & \Rightarrow \quad |x-4| < \frac{15}{2}\varepsilon \\
& \Rightarrow \quad \frac{2}{3}\frac{|x-4|}{5} < \varepsilon \\
& \Rightarrow \quad \frac{2}{3}\frac{|x-4|}{|x+2|} < \varepsilon \text{ since } \frac{1}{|x+2|} < \frac{1}{5} \\
& \Rightarrow \quad \frac{2}{3}\left|\frac{4-x}{x+2}\right| < \varepsilon \\
& \Rightarrow \quad \left|\frac{8-2x)}{3(x+2)}\right| < \varepsilon \\
& \Rightarrow \quad \left|\frac{12-2(x+2)}{3(x+2)}\right| < \varepsilon \\
& \Rightarrow \quad \left|f(x) - \frac{2}{3}\right| < \varepsilon
\end{aligned}$$

Therefore, $|x-4| < \delta \Rightarrow \left|f(x) - \frac{2}{3}\right| < \varepsilon$. ■

■ **Example 3.8** Show that $\lim_{x\to 1} f(x) = 5$ by using $\varepsilon - \delta$ definition, where $f(x) = 4 + \sqrt{x}$.
Proof: Let $\varepsilon > 0$ be given. We must find $\delta > 0$ such that

$$|x-1| < \delta \Rightarrow |f(x) - 5| < \varepsilon$$

Consider

$$\begin{aligned}
|f(x) - 5| < \varepsilon \quad & \Leftrightarrow \quad |4 + \sqrt{x} - 5| < \varepsilon \\
& \Leftrightarrow \quad |\sqrt{x} - 1| < \varepsilon \\
& \Leftrightarrow \quad |\frac{x-1}{\sqrt{x}+1}| < \varepsilon \\
& \Leftrightarrow \quad \frac{|x-1|}{|\sqrt{x}+1|} < \varepsilon
\end{aligned}$$

Let choose $\delta_1 = 1$. Then,

$$\begin{aligned}
|x-1| < \delta_1 = 1 \quad & \Rightarrow \quad -1 < x-1 < 1 \\
& \Rightarrow \quad 0 < x < 2 \\
& \Rightarrow \quad 0 < \sqrt{x} < \sqrt{2} \\
& \Rightarrow \quad 1 < \sqrt{x}+1 < \sqrt{2}+1 \\
& \Rightarrow \quad \frac{1}{\sqrt{2}+1} < \frac{1}{\sqrt{x}+1} < \frac{1}{1} \\
& \Rightarrow \quad \frac{|x-1|}{\sqrt{2}+1} < \frac{|x-1|}{|\sqrt{x}+1|} < \frac{|x-1|}{1} = |x-1|
\end{aligned}$$

Let for some values of x we have $|x-1| < \varepsilon$. So choose $\delta_2 = \varepsilon$.
Now choose $\delta = \min\{\delta_1, \delta_2\} = \{1, \varepsilon\}$.

$$\begin{aligned}
0 < |x-1| < \delta \quad & \Rightarrow \quad |x-1| < \varepsilon \\
& \Rightarrow \quad \frac{|x-1|}{|\sqrt{x}+1|} < \varepsilon \text{ since } \frac{|x-1|}{|\sqrt{x}+1|} < \frac{|x-1|}{1} = |x-1| \\
& \Rightarrow \quad |\frac{x-1}{\sqrt{x}+1}| < \varepsilon \\
& \Rightarrow \quad |\sqrt{x} - 1| < \varepsilon \\
& \Rightarrow \quad |4 + \sqrt{x} - 5| < \varepsilon \\
& \Rightarrow \quad |f(x) - 5| < \varepsilon
\end{aligned}$$

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Therefore, $|x-4|<\delta \Rightarrow |f(x)-\frac{2}{3}|<\varepsilon$. ■

3.2 Basic Limit Theorems

If $\lim_{x\to \mathbf{a}} f(x) = L$ and $\lim_{x\to \mathbf{a}} g(x) = M$, then

1. Limit of a constant function; that is, $f(x) = c$ for all x for any constant c

$$\lim_{x\to \mathbf{a}} c = c$$

This means the graph of the constant function f is a horizontal line. That is no matter what the value of x, $f(x)$ is always c. Thus as we approach $x = c$ from either the left or the right, we hit the line $y = c$ at height of c.

■ **Example 3.9** $\lim_{x\to 2} 5 = 5$ ■

2. Limit of identity function; that is, $f(x) = x$ for all x

$$\lim_{x\to \mathbf{a}} x = \mathbf{a}$$

The graph of the function is a straight line. As we approach the point $x = a$ from the left and the right, the function approaches the value **a**.

■ **Example 3.10** $\lim_{x\to 4} x = 4$ ■

3. Constant Multiple Rule

$$\begin{aligned} \lim_{x\to \mathbf{a}} cf(x) &= c\lim_{x\to \mathbf{a}} f(x) \\ &= cL \text{ for any constant c} \end{aligned}$$

■ **Example 3.11** $\lim_{x\to 2} 3x = 3\lim_{x\to 2} x = 3(2) = 6$ ■

4. Sum–Difference Rule

$$\begin{aligned} \lim_{x\to \mathbf{a}} (f(x) \pm g(x)) &= \lim_{x\to \mathbf{a}} f(x) \pm \lim_{x\to \mathbf{a}} g(x) \\ &= L \pm M \end{aligned}$$

■ **Example 3.12** $\lim_{x\to 2}(x+6) = \lim_{x\to 2} x + \lim_{x\to 2} 6 = 3(2)+6 = 12$ ■

5. Product Rule

$$\begin{aligned} \lim_{x\to \mathbf{a}} (f(x)\cdot g(x)) &= \lim_{x\to \mathbf{a}} f(x) \cdot \lim_{x\to \mathbf{a}} g(x) \\ &= L\cdot M \end{aligned}$$

■ **Example 3.13**

$$\begin{aligned} \lim_{x\to 2} x^2 &= \lim_{x\to 2} x\cdot x \\ &= \lim_{x\to 2} x \cdot \lim_{x\to 2} x \\ &= 2\cdot 2 = 4 \end{aligned}$$

■

■ **Example 3.14**

$$\begin{aligned} \lim_{x\to 2}(x+1)(x^2-2) &= \lim_{x\to 2}(x+1)\cdot \lim_{x\to 2}(x^2-2) \\ &= 3\cdot 2 = 6 \end{aligned}$$

■

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6. Quotient Rule

$$\lim_{x\to\mathbf{a}}\frac{f(x)}{g(x)} = \frac{\lim_{x\to\mathbf{a}} f(x)}{\lim_{x\to\mathbf{a}} g(x)} = \frac{L}{M} \text{ provided that } M \neq 0$$

7. Power Rule
$\lim_{x\to\mathbf{a}}(f(x))^n = (\lim_{x\to\mathbf{a}} f(x))^n = L^n$, n is any real number

■ **Example 3.15** $\lim_{x\to 2}(3x+2)^2 = (\lim_{x\to 2} 3x+2)^2 = 8^2 = 64$ ■

Although the above basic limit theorems are stated for only two functions f and g, the result will still be true for a finite number of functions.

■ **Example 3.16** Find the $\lim_{x\to 1}(5x^2+3x+1)$.
Solution:

$$\begin{aligned}\lim_{x\to 1}(5x^2+3x+1) &= \lim_{x\to 1} 5x^2 + \lim_{x\to 1} 3x + \lim_{x\to 1} 1 \\ &= 5(\lim_{x\to 1} x)^2 + 3\lim_{x\to 1} x + \lim_{x\to 1} 1 \\ &= 5(1)^2 + 3(1) + 1 \\ &= 9\end{aligned}$$

■

■ **Example 3.17** Find the $\lim_{x\to 2}\frac{9x^5+10}{7x^2+2x+1}$.
Solution:

$$\begin{aligned}\lim_{x\to 1}\frac{9x^2+10}{7x^2+2x+1} &= \frac{\lim_{x\to 2} 9x^5+10}{\lim_{x\to 2} 7x^2+2x+1} \\ &= \frac{9(\lim_{x\to 2} x)^5 + \lim_{x\to 2} 10}{7(\lim_{x\to 2} x)^2 + 2\lim_{x\to 2} x + \lim_{x\to 2} 1} \\ &= \frac{9(2)^5+10}{7(2)^2+2(2)+1} \\ &= \frac{298}{33}\end{aligned}$$

■

From the above examples, we can have the following formulas to evaluate limits of a polynomial and rational functions.
If $f(x)$ and $g(x)$ are polynomials, then

$$\lim_{x\to\mathbf{a}} f(x) = f(\mathbf{a}) \text{ and } \lim_{x\to\mathbf{a}}\frac{f(x)}{g(x)} = \frac{f(\mathbf{a})}{g(\mathbf{a})}, g(\mathbf{a}) \neq 0$$

In both cases you simply substitute the value of **a** in to the equation.

■ **Example 3.18** Find the $\lim_{x\to 1}\frac{x+3}{x-1}$.
Solution: Since $\lim_{x\to 1}(x-1) = 0$, the quotient rule for limits does not apply here. When the denominator of the given rational function approaches zero, while the numerator does not, we can conclude that the limit does not exist.
When both the numerator and the denominator of the given rational function approaches zero, simplify the function algebraically in order to find the desired limits. The limits of $f(x)$ as x approaches **a**, depends on the values of $f(x)$ as x becomes close to a, but we exclude $x = \mathbf{a}$. ■

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■ **Example 3.19** Evaluate $\lim\limits_{x\to 2} \frac{x^2-x-2}{x-2}$.
Solution: When substitute 2 in to the function, then both the numerator and denominator becomes zero, hence $f(2)$ is meaningless. Therefore, straight substitution does not yield the limit of the function f at $x=2$. Simplifying the function, we get

$$f(x) = \frac{x^2-x-2}{x-2} = \frac{(x-2)(x+1)}{(x-2)} = (x+1)$$

Thus,

$$\lim_{x\to 2} \frac{x^2-x-2}{x-2} = \lim_{x\to 2}(x+1) = 3$$

■

Remark 3.2.1 Even after expressing f in the form of $x+1$, we do not think what $x+1$ is when $x=2$ but rather that of what $x+1$ approaches as x tends to 2.

■ **Example 3.20** Evaluate $\lim\limits_{x\to -2} \frac{x^2}{|x|+3}$.
Solution: Observe that $\frac{x^2}{|x|+3}$ is the quotient of x^2 and $|x|+3$, whose limits at -2. We know that

$$\lim_{x\to -2} x^2 = 4 \text{ and } \lim_{x\to -2}(|x|+3) = 5$$

We conclude from the quotient rule that

$$\begin{aligned}\lim_{x\to -2} \frac{x^2}{|x|+3} &= \frac{\lim\limits_{x\to -2} x^2}{\lim\limits_{x\to -2}(|x|+3)} \\ &= \frac{4}{5}\end{aligned}$$

■

■ **Example 3.21** Show that $\lim\limits_{x\to 0} \frac{1}{x}$ does not exist.
Solution: Suppose that $\lim\limits_{x\to 0} \frac{1}{x}$ exists, and let $L = \lim\limits_{x\to 0} \frac{1}{x}$. Since $1 = x(\frac{1}{x})$, by using product rule

$$\begin{aligned}1 &= \lim_{x\to 0} 1 \\ &= \lim_{x\to 0}(x\frac{1}{x}) \\ &= \lim_{x\to 0} x \cdot \lim_{x\to 0} \frac{1}{x} \\ &= 0 \cdot L \\ &= 0\end{aligned}$$

Which is obvious false. Therefore, $\lim\limits_{x\to 0} \frac{1}{x}$ does not exist. ■

■ **Example 3.22** Find $\lim\limits_{x\to -2} \frac{x^3+2x^2-x-2}{x^2-4}$.
Solution: Since $\lim\limits_{x\to -2}(x^2-4)$, we can not apply the quotient rule to this function in its original form. However, since $x^3+2x^2-x-2 = (x+2)(x^2-1)$ and $x^2-4 = (x+2)(x-2)$, we have

$$\begin{aligned}\lim_{x\to -2} \frac{x^3+2x^2-x-2}{x^2-4} &= \frac{\lim\limits_{x\to -2}(x+2)(x^2-1)}{(x+2)(x-2)} \\ &= \lim_{x\to -2} \frac{x^2-1}{x-2} \\ &= \frac{(-2)^2-1}{-2-2} \\ &= -\frac{3}{4}\end{aligned}$$

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■

■ **Example 3.23** Evaluate $\lim_{x\to 9} \frac{x(\sqrt{x}-3)}{x-9}$.
Since $\lim_{x\to 9}(x-9) = 0$, the quotient rule can not be applied directly. However, if we factor the denominator, then we can cancel terms and the quotient rule

$$\begin{aligned}
\lim_{x\to 9} \frac{x(\sqrt{x}-3)}{x-9} &= \lim_{x\to 9} \frac{x(\sqrt{x}-3)}{(\sqrt{x}-3)(\sqrt{x}+3)} \\
&= \lim_{x\to 9} \frac{\sqrt{x}}{\sqrt{x}+3} \\
&= \frac{9}{\sqrt{9}+3} \\
&= \frac{3}{2}
\end{aligned}$$

■

Theorem 3.2.2 Squeezing Principle
Suppose that $f(x) \leq h(x) \leq g(x)$ for all $x \neq a$ in some neighborhood. Suppose also that

$$\lim_{x\to a} f(x) = k = \lim_{x\to a} g(x)$$

Then, we also have $\lim_{x\to a} h(x) = k$.

Theorem 3.2.3 Important Limit Theorems

$$\begin{aligned}
&\lim_{x\to 0} \frac{\sin x}{x} = 1 && \lim_{x\to 0} \frac{1-\cos x}{x} = 0 \\
&\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2} && \lim_{x\to 0} \frac{\tan x}{x} = 1 \\
&\lim_{x\to\infty} (1+\frac{1}{x})^x = e && \lim_{x\to 0^+} (1+x)^{\frac{1}{x}} = e \\
&\lim_{x\to 0} \frac{e^x-1}{x} = 1 && \lim_{x\to 1} \frac{x-1}{\ln x} = 1
\end{aligned}$$

■ **Example 3.24** Show that $\lim_{x\to 0} \frac{\sin x}{x} = 1$
Solution: Construct a circle with center at O and radius $OA = OD = 1$, as in figure below. Choose point B on A extended and point C on OD so that lines BD and AC are perpendicular to OD. It is geometrically

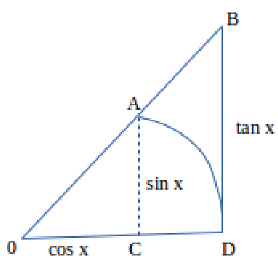

Figure 3.4: Circle with center O

evident that

$$\text{Area of } \triangle OAC < \text{Area of a sector } OAD < \text{Area of } \triangle OBD \tag{3.1}$$

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From this

$$\frac{1}{2}\sin x\cos x < \frac{1}{2}x < \frac{1}{2}\tan x \tag{3.2}$$

Divide equation (3.2) both sides by $\frac{1}{2}\sin x$, then we have

$$\cos x < \frac{x}{\sin x} < \frac{1}{\cos x}$$
$$\Rightarrow \cos x < \frac{\sin x}{x} < \frac{1}{\cos x}$$

As $x \to 0$, $\cos x \to 1$ and $\frac{1}{\cos x} \to 1$, by squeezing theorem it follows that

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

■

Remark 3.2.4
- $\lim_{x\to 0}\frac{\sin kx}{x} = k$ and $\lim_{x\to 0}\frac{\tan kx}{x} = k$ for any constant k.
- $\lim_{x\to 0}\frac{\sin^n x}{x} = 1$ for any positive integer n.

■ **Example 3.25** Show that

$$\lim_{x\to 0}\frac{\cos x - 1}{x} = 0$$

Solution: Notice that $\lim_{x\to 0} x = 0$, so we can not apply the quotient rule directly. However

$$\begin{aligned}\lim_{x\to 0}\frac{\cos x - 1}{x} &= \lim_{x\to 0}\left(\frac{\cos x - 1}{x}\right)\cdot\left(\frac{\cos x + 1}{\cos x + 1}\right)\\ &= \lim_{x\to 0}\frac{\cos^2 x - 1}{x(\cos x + 1)}\\ &= \lim_{x\to 0}\frac{-\sin^2 x}{x(\cos x + 1)}\\ &= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)\left(\frac{-\sin x}{\cos x + 1}\right)\end{aligned}$$

Since $\lim_{x\to 0}\frac{\sin x}{x} = 1$. Furthermore, $\lim_{x\to 0}\frac{-\sin x}{\cos x+1} = \frac{0}{1+1} = 0$.
By the sum and the quotient rules. Thus the product rule tells us that

$$\begin{aligned}\lim_{x\to 0}\frac{\cos x - 1}{x} &= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)\left(\frac{-\sin x}{\cos x + 1}\right)\\ &= 1\cdot 0\\ &= 0\end{aligned}$$

■

■ **Example 3.26** Evaluate $\lim_{x\to 1}\frac{\sqrt{x}-x^2}{1-\sqrt{x}}$.

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Solution:

$$\begin{aligned}
\lim_{x\to 1}\frac{\sqrt{x}-x^2}{1-\sqrt{x}} &= \lim_{x\to 1}\frac{\sqrt{x}-x^2}{1-\sqrt{x}}\left(\frac{1+\sqrt{x}}{1+\sqrt{x}}\right)\\
&= \lim_{x\to 1}\frac{\sqrt{x}+x-x^2x^2\sqrt{x}}{1-x}\\
&= \lim_{x\to 1}\frac{\sqrt{x}-x^2\sqrt{x}+x-x^2}{1-x}\\
&= \lim_{x\to 1}\frac{\sqrt{x}(1-x^2)+x(1-x)}{1-x}\\
&= \lim_{x\to 1}\frac{\sqrt{x}(1-x)(1+x)+x(1-x)}{1-x}\\
&= \lim_{x\to 1}\frac{(1-x)(\sqrt{x}(1+x)+x)}{1-x}\\
&= \lim_{x\to 1}(\sqrt{x}(1+x)+x)\\
&= 3
\end{aligned}$$

■

■ **Example 3.27** Is there a number **a** such that

$$\lim_{x\to -2}\frac{3x^2+\mathbf{a}x+\mathbf{a}+3}{x^2+x-2}$$

exist? If so find the value of **a** and the value of the limit.
Solution: Since the limit of $\lim_{x\to -2}\frac{3x^2+\mathbf{a}x+\mathbf{a}+3}{x^2+x-2}$ exist, the numerator must be the factor of $(x+2)$; that is,

$$\lim_{x\to -2}\frac{3x^2+\mathbf{a}x+\mathbf{a}+3}{x^2+x-2}=\lim_{x\to -2}\frac{3x^2+\mathbf{a}x+\mathbf{a}+3}{(x-1)(x+2)}$$

Thus

$$\begin{aligned}
&3x^2+\mathbf{a}x+\mathbf{a}+3=(bx+c)(x+2)\\
&\Rightarrow 3x^2+\mathbf{a}x+\mathbf{a}+3=bx^2+(2b+c)x+2c\\
&\Rightarrow b=3, 2b+c=\mathbf{a} \text{ and } \mathbf{a}+3=2c\\
&\Rightarrow \mathbf{a}=15, b=3 \text{ and } c=9
\end{aligned}$$

So,

$$\begin{aligned}
\lim_{x\to -2}\frac{3x^2+\mathbf{a}x+\mathbf{a}+3}{x^2+x-2} &= \lim_{x\to -2}\frac{(3x+9)(x+2)}{(x-1)(x-2)}\\
&= \lim_{x\to -2}\frac{3x+9}{x-1}\\
&= -1
\end{aligned}$$

■

■ **Example 3.28** Show that $\lim_{x\to 0}x^3\sin\frac{1}{x}=0$.
Solution: First notice that we can not use

$$\lim_{x\to 0}x^3\sin\frac{1}{x}=\lim_{x\to 0}x^3\lim_{x\to 0}\sin\frac{1}{x}$$

because $\lim_{x\to 0}\sin\frac{1}{x}$ does not exist.
However, since

$$-1\le \sin\frac{1}{x}\le 1$$

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We have, $-x^3 \leq x^3 \sin\frac{1}{x} \leq x^3$. We know that

$$\lim_{x\to 0} x^3 = 0 \text{ and } \lim_{x\to 0}(-x)^3 = 0$$

Taking $f(x) = -x^3$, $g(x) = x^3 \sin\frac{1}{x}$, and $h(x) = x^3$ in the squeezing theorem, we obtain

$$\lim_{x\to 0} x^3 \sin\frac{1}{x} = 0$$

■

3.3 One sided Limit

Definition 3.3.1

1. Let f be a function which is defined at every number in some open interval $(\mathbf{a},\mathbf{c})$. Then the limit of $f(x)$, as x approaches $\mathbf{a}$ from the right, is L, written

$$\lim_{x\to \mathbf{a}^+} f(x) = L$$

if for any $\varepsilon > 0$, however small, there exist a δ such that, $|f(x(-L| < \varepsilon$ whenever $0 < x - \mathbf{a} < \delta$.

2. Let f be a function which is defined at every number in some open interval $(\mathbf{d},\mathbf{a})$. Then the limit of $f(x)$, as x approaches $\mathbf{a}$ from the left, is L, written

$$\lim_{x\to \mathbf{a}^-} f(x) = L$$

if for any $\varepsilon > 0$, however small, there exist a δ such that, $|f(x(-L| < \varepsilon$ whenever $-\delta < x - \mathbf{a} < 0$.

3. The limit of a function exists at $x = a$ if and only if

$$\lim_{x\to \mathbf{a}^-} f(x) = L = \lim_{x\to \mathbf{a}^+} f(x)$$

■ **Example 3.29** Find the limit of $f(x)$ as x approaches 2 from the left and the right, where $f(x) = x - 3$ and determine whether the limit of the function exist at $x = 2$ or not.
Solution: First we have to evaluate one sided limits from both sides of $x = 2$

$$\begin{aligned} \lim_{x\to 2^-} f(x) &= -1 \\ \lim_{x\to 2^+} f(x) &= -1 \end{aligned}$$

Therefore, $\lim_{x\to 2} f(x) = -1$. ■

■ **Example 3.30** Find $\lim_{x\to 1} f(x)$, where $f(x) = \begin{cases} x+1, & x<1 \\ 3-x, & x>1 \end{cases}$
Solution: Evaluate the limit as x approaches 1. From the graph we can see that

$$\begin{aligned} \lim_{x\to 1^-} f(x) &= 2 \\ \lim_{x\to 1^+} f(x) &= 2 \end{aligned}$$

Therefore, $\lim_{x\to 1} f(x) = 2$. ■

■ **Example 3.31** Find $\lim_{x\to 1} f(x)$, where $f(x) = \frac{|x|}{x}$.

Solution: Notice that $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x<0 \end{cases}$ and $\frac{|x|}{x} = \begin{cases} 1, & x>0 \\ -1, & x<0 \end{cases}$
Then the left side limit can be found as

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^-} -1 = -1$$

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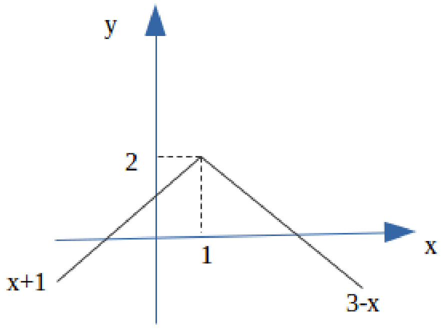

Figure 3.5: $f(x) = x+1$ if $x < 1$ and $f(x) = 3-x$ if $x > 1$

and the right side limit can be found as

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} 1 = 1$$

Therefore, $\lim_{x\to 0} f(x)$ does not exist since the left side and right side limits are not the same; that is, $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$. Graphically, it is shown in the figure below. The value of $\frac{|x|}{x}$ approach different

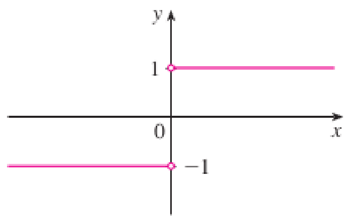

Figure 3.6: $f(x) = \frac{|x|}{x}$

numbers as x approaches 0 from different sides, so $\lim_{x\to 0} \frac{|x|}{x}$ does not exist. ■

3.4 Infinite Limit, Limit at infinity and Asymptotes

3.4.1 Infinite Limit and Vertical Asymptote

Definition 3.4.1 Let f be a function defined in an interval containing $\mathbf{a}$, with the possible exception of $\mathbf{a}$ it self. Then,

1. $\lim_{x\to \mathbf{a}^+} f(x) = \infty$, if for every number $M > 0$ there is some $\delta > 0$ such that $0 < x - \mathbf{a} < \delta$, then $f(x) > M$.
2. $\lim_{x\to \mathbf{a}^+} f(x) = -\infty$, if for every number $M < 0$ there is some $\delta > 0$ such that $0 < x - \mathbf{a} < \delta$, then $f(x) < M$.
3. $\lim_{x\to \mathbf{a}^-} f(x) = \infty$, if for every number $M > 0$ there is some $\delta > 0$ such that $-\delta < x - \mathbf{a} < 0$, then $f(x) > M$.
4. $\lim_{x\to \mathbf{a}^-} f(x) = -\infty$, if for every number $M < 0$ there is some $\delta > 0$ such that $-\delta < x - \mathbf{a} < 0$, then $f(x) < M$.

■ **Example 3.32** Let $f(x) = \frac{1}{x}$, show that $\lim_{x\to 0^+} f(x) = \infty$ and $\lim_{x\to 0^+} f(x) = \infty$.

Solution: For every $M > 0$, we can find $\delta > 0$ such that

$$\begin{aligned} 0 < x - 0 < \delta &\Leftrightarrow f(x) > M \\ 0 < x < \delta &\Leftrightarrow \frac{1}{x} > M \\ &\Leftrightarrow x < \frac{1}{M} \end{aligned}$$

Now choose $\delta = \frac{1}{M}$. Thus

$$\begin{aligned} 0 < x - 0 < \delta &\Leftrightarrow \frac{1}{x} > \frac{1}{\delta} = M \\ &\Leftrightarrow f(x) > M \end{aligned}$$

Therefore, $\lim\limits_{x \to 0^+} f(x) = \infty$ and for every $M < 0$, we can find $\delta > 0$ such that

$$\begin{aligned} -\delta < x - 0 < 0 &\Leftrightarrow f(x) < M \\ -\delta < x < 0 &\Leftrightarrow \frac{1}{x} < M \\ &\Leftrightarrow x > \frac{1}{M} \end{aligned}$$

Now choose $\delta = -\frac{1}{M}$. Thus

$$\begin{aligned} -\delta < x - 0 < 0 &\Leftrightarrow \frac{1}{x} < -\frac{1}{\delta} = M \\ &\Leftrightarrow f(x) < M \end{aligned}$$

Therefore, $\lim\limits_{x \to 0^-} f(x) = \infty$. ■

■ **Example 3.33** Show that $\lim\limits_{x \to \frac{1}{2}^-} f(x) = -\infty$, where $f(x) = \frac{5x+1}{2x-1}$.

Solution: For every $M < 0$, there is a number $\delta > 0$ such that

$$-\delta < x - \frac{1}{2} < 0 \Rightarrow \frac{5x+1}{2x-1} < M$$

Consider

$$\begin{aligned} \frac{5x+1}{2x-1} < M &\Leftrightarrow \frac{2x-1}{5x+1} > \frac{1}{M} \\ &\Leftrightarrow \frac{2(x-\frac{1}{2})}{5x+1} > \frac{1}{M} \end{aligned}$$

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Let choose $\delta_1 \leq \frac{1}{5}$, then

$$\begin{aligned}
-\frac{1}{5} < -\delta < x - \frac{1}{2} < 0 \quad &\Leftrightarrow \quad -\frac{1}{5} < x - \frac{1}{2} < 0 \\
&\Leftrightarrow \quad -\frac{1}{5} < x - \frac{1}{2} < 0 \\
&\Leftrightarrow \quad \frac{3}{10} < x < \frac{1}{2} \\
&\Leftrightarrow \quad \frac{15}{10} < 5x < \frac{5}{2} \\
&\Leftrightarrow \quad \frac{25}{10} < 5x+1 < \frac{7}{2} \\
&\Leftrightarrow \quad \frac{2}{7} < \frac{1}{5x+1} < \frac{10}{25} \\
&\Leftrightarrow \quad \frac{4}{7} < \frac{2}{5x+1} < \frac{20}{25} \\
&\Leftrightarrow \quad \frac{20}{25}(x-\frac{1}{2}) < \frac{2}{5x+1}(x-\frac{1}{2}) < \frac{4}{7}(x-\frac{1}{2}) \\
&\quad \quad \text{since } x - \frac{1}{2} < 0
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{2(x-\frac{1}{2})}{5x+1} > \frac{1}{M} \quad &\Leftrightarrow \quad \frac{1}{M} < \frac{2(x-\frac{1}{2})}{5x+1} < \frac{4}{7}(x-\frac{1}{2})\frac{1}{5x+1} \\
&\Leftrightarrow \quad \frac{1}{M} < \frac{4}{7}(x-\frac{1}{2})\frac{1}{5x+1} \\
&\Leftrightarrow \quad \frac{1}{M} < \frac{4}{7}\frac{x-\frac{1}{2}}{5x+1} \\
&\Leftrightarrow \quad \frac{7}{4M} < x - \frac{1}{2}
\end{aligned}$$

Choose $\delta_2 = -\frac{7}{4M}$. Now choose $\delta = \min\{\delta_1, \delta_2\} = \{\frac{1}{5}, -\frac{7}{4M}\}$. Thus,

$$\begin{aligned}
-\delta < x - \frac{1}{2} < 0 \quad &\Rightarrow \quad \frac{7}{4M} < x - \frac{1}{2} < 0 \\
&\Rightarrow \quad \frac{7}{4M} < x - \frac{1}{2} \\
&\Rightarrow \quad \frac{7}{2M} < 2x - 1 \\
&\Rightarrow \quad \frac{7}{2}\frac{1}{2x-1} < M \\
&\Rightarrow \quad (5x+1)\frac{1}{2x-1} < \frac{7}{2}\frac{1}{2x-1} < M \\
&\Rightarrow \quad \frac{5x+1}{2x-1} < M \\
&\Rightarrow \quad f(x) < M
\end{aligned}$$

Therefore, $\lim_{x \to \frac{1}{2}^-} f(x) = -\infty$. ∎

Definition 3.4.2 The line $x = \mathbf{a}$ is called a vertical asymptote of the graph of $y = f(x)$ if any one of the following limits holds true

- $\lim_{x \to \mathbf{a}^-} f(x) = \pm\infty$

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- $\lim_{x \to \mathbf{a}^+} f(x) = \pm\infty$
- $\lim_{x \to \mathbf{a}} f(x) = \pm\infty$

■ **Example 3.34** The line $x = 0$ is a vertical asymptote of the graph $f(x) = \frac{1}{x}$ since $\lim_{x \to 0^-} \frac{1}{x} = -\infty$ and $\lim_{x \to 0^+} \frac{1}{x} = \infty$. ■

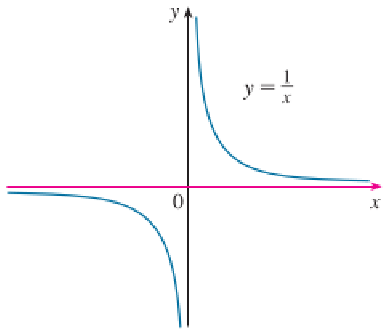

Figure 3.7: $y = \frac{1}{x}$

Definition 3.4.3 The point $x = x_0$ is a hole to the graph of a rational function f if and only $\lim_{x \to x_0} f(x)$ exist and $f(x_0)$ does not defined.

■ **Example 3.35** Determine the vertical asymptote or a hole to the graph of $f(x) = \frac{x^2-1}{x^2-x}$, if it exists.
Solution: The zeros of $x^2 - x = 0$ are $x = 0$ or $x = 1$. So, f is not defined at $x = 0$ and $x = 1$. Now, find the limit of $f(x)$ at $x = 0$ and $x = 1$.

$$\lim_{x \to 1} \frac{x^2 - 1}{x^2 - x} = \lim_{x \to 1} \frac{x+1}{x} = 2$$

Thus, $f(x)$ has a hole at $x = 1$ and

$$\lim_{x \to 0} \frac{x^2 - 1}{x^2 - x} = \pm\infty$$

Thus, $f(x)$ has a vertical asymptote at $x = 0$. ■

3.4.2 Limit at Infinity and Horizontal Asymptotes

Definition 3.4.4
1. Let f be a function defined in an interval $(\mathbf{a}, \infty)$, then $\lim_{x \to \infty} f(x) = L$ if and only if for all $\varepsilon > 0$, there exist $M > 0$ such that if $x > M$, then $|f(x) - L| < \varepsilon$.
2. Let f be a function defined in an interval $(-\infty, \mathbf{a})$, then $\lim_{x \to -\infty} f(x) = L$ if and only if for all $\varepsilon > 0$, there exist $N < 0$ such that if $x < N$, then $|f(x) - L| < \varepsilon$.

Definition 3.4.5 The line $y = L$ is called a horizontal asymptote of the graph of $y = f(x)$ if either $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$.

■ **Example 3.36** Consider the behavior of the function $f(x) = \frac{1}{x}, x \neq 0$ when x tends to infinity.
Solution: As x gets larger and larger and continuous to grow without bound, the corresponding values of f get closer and closer to 0 and eventually tend to 0. The values of $f(x)$ becomes closer and closer to zero as x approaches to infinity

$$\lim_{x \to \infty} f(x) = 0$$

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This is read as 'the limit of f is zero as x tends to positive infinitive'.
Similarly the limit of f is zero as x tends to negative infinitive.

$$\lim_{x\to-\infty} f(x) = 0$$

Since $\lim_{x\to\infty} \frac{1}{x} = 0$ and $\lim_{x\to-\infty} \frac{1}{x} = 0$, the line $y = 0$ is the horizontal asymptote of $f(x) = \frac{1}{x}$. ■

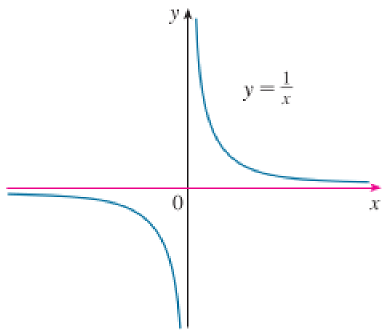

Figure 3.8: $y = \frac{1}{x}$

■ **Example 3.37** Evaluate $\lim_{x\to\infty} \frac{3x-1}{x+2}$.
Solution: Let us divide both the numerator and denominator by x

$$\frac{3x-1}{x+2} = \frac{3-\frac{1}{x}}{1+\frac{2}{x}}, x \neq 0, x \neq -2$$

Since we are concerned with the behavior of $\frac{3x-1}{x+2}$ for sufficiently large values of x. Thus we have

$$\begin{aligned} \lim_{x\to\infty} \frac{3x-1}{x+2} &= \lim_{x\to\infty} \frac{3-\frac{1}{x}}{1+\frac{2}{x}} \\ &= \frac{3-\lim_{x\to\infty}\frac{1}{x}}{1+\lim_{x\to\infty}\frac{2}{x}} \end{aligned}$$

Therefore, $\lim_{x\to\infty} \frac{3x-1}{x+2} = 3$. ■

■ **Example 3.38** Find the vertical and horizontal asymptotes of the graph of $f(x) = \frac{1}{x-3}$.
Solution:

- To find Vertical Asymptotes
  Notice that if

$$\begin{aligned} \lim_{x\to\mathbf{a}^-} f(x) &= \pm\infty \\ \lim_{x\to\mathbf{a}^+} f(x) &= \pm\infty \text{ or} \\ \lim_{x\to\mathbf{a}} f(x) &= \pm\infty \end{aligned}$$

  Then the line $x = \mathbf{a}$ is a vertical asymptote of $f(x)$.
  Now, find $\lim_{x\to 3^-} \frac{1}{x-3} = -\infty$ and also $\lim_{x\to 3^+} \frac{1}{x-3} = \infty$. Thus, the line $x = 3$ is the vertical asymptote.
- To find Horizontal Asymptotes
  If either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then the line $y = L$ is called a horizontal asymptote of the graph of $y = f(x)$.
  Now, find $\lim_{x\to\pm\infty} \frac{1}{x-3} = 0$ is horizontal asymptote.

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■

Theorem 3.4.1 The limit of a function (if exists) is unique.

3.5 Continuity of a Function

Definition 3.5.1 A function f is said to be continuous at $\mathbf{a}$ if $\lim_{x\to\mathbf{a}} f(x) = f(\mathbf{a})$

Definition 3.5.2 If f is continuous at $\mathbf{a}$, then

1. $f(\mathbf{a})$ is defined
2. $\lim_{x\to\mathbf{a}} f(x)$ exist
3. $\lim_{x\to\mathbf{a}} f(x) = f(\mathbf{a})$.

If f is not continuous at $\mathbf{a}$, then we say that f is discontinuous at $\mathbf{a}$.

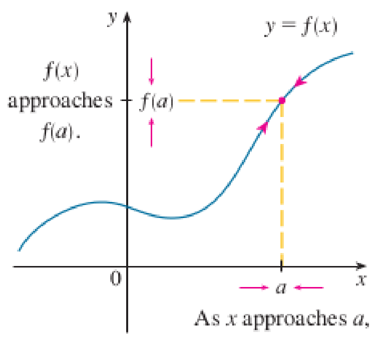

Figure 3.9: Continuity of $f(x)$ as x approaches $\mathbf{a}$

If $\mathbf{a}$ is in the domain of the rational function f, then $\lim_{x\to\mathbf{a}} f(x) = f(\mathbf{a})$. Thus any rational function is continuous at every point in its domain.

■ **Example 3.39** Show that the polynomial function $f(x) = 3x^2 - x + 5$ is continuous at $x = 1$.
Solution: First we have to check whether the above three condition of continuity are satisfied or not

1. $f(1)$ is defined
2. $\lim_{x\to 1} f(x)$ exist
3. $\lim_{x\to 1} f(x) = f(1)$

Therefore, the function $f(x) = 3x^2 - x + 5$ is continuous at $x = 1$. ■

■ **Example 3.40** Show that $f(x) = \frac{x^2-1}{x-1}$ is not continuous at $x = 1$.
Solution:

1. $f(1)$ is not defined
2. $\lim_{x\to 1} f(x)$ exist

Even if the limit of the function exists, since it is not defined at $x = 1$, the function $f(x)$ is discontinuous at $x = 1$. ■

■ **Example 3.41** Find the points of discontinuity of $f(x) = \frac{x^2-x-6}{x-2}$.
Solution: The three condition for continuity are satisfied for any values of $x \in R$ except at $x = 2$. Therefore, $x = 2$ is the only discontinuity point for $f(x) = \frac{x^2-x-6}{x-2}$. ■

3.5.1 One Sided Continuity

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Definition 3.5.3 1. f is said to be continuous from the right at $\mathbf{a}$ if $\lim_{x\to\mathbf{a}^+} f(x) = f(\mathbf{a})$.
2. f is said to be continuous from the left at $\mathbf{a}$ if $\lim_{x\to\mathbf{a}^-} f(x) = f(\mathbf{a})$.

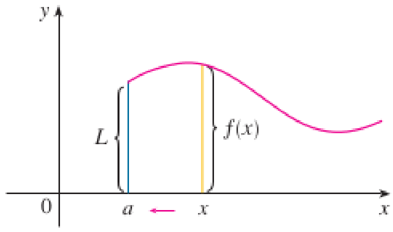

Figure 3.10: Continuity of f as x approaches to a from the right

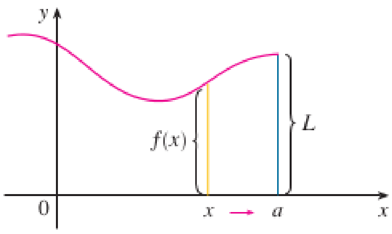

Figure 3.11: Continuity of f as x approaches to a from the left

■ **Example 3.42** Show that f is continuous from the right at 0, but not continuous from the left at 0, where $f(x)$ defined by

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

Solution: Since $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 1 = 1$ and $f(0) = 1$, it follows that $\lim_{x\to 0^+} f(x) = f(0)$
Hence f is continuous from the right at zero. On the other hand, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} -1 = -1$ and $f(0) = 1$.
since $\lim_{x\to 0^-} f(x) \neq f(0)$, f is not continuous from the left at zero. ■

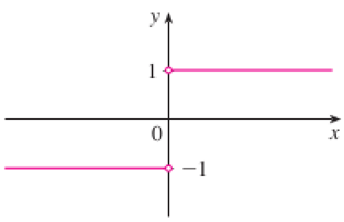

Figure 3.12:

■ **Example 3.43** Show that the function defined by

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

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is continuous from the right at 0 but not continuous from the left at 0.
Solution: $\lim_{x\to 0^+} f(x) = 1 = f(0)$ because of this f is continuous from the right at 0, f is not continuous from the left at 0 since $\lim_{x\to 0^-} f(x) = 0 \neq f(0)$. ■

Definition 3.5.4 A function f is said to be

1. Continuous on (a,b) if f is continuous at each point in the open interval (a,b).
2. Continuous function if f is continuous over its domain.
3. Continuous in the closed interval $[a,b]$ if f is continuous at each point in the open interval (a,b) and $x = a$ from the right and at $x = b$ from the left.

■ **Example 3.44** Show that the function $f(x) = \frac{x+5}{x-2}$ is continuous on the open interval $(-3,2)$.
Solution: The three conditions for continuity are satisfied for for any value of x between -3 and 2. Therefore, the function is continuous in the open interval $(-3,2)$. However, the function is not continuous for the closed interval $[-3,2]$, since $f(x)$ is discontinuous at $x = 2$. ■

■ **Example 3.45** Show that $f(x) = \sqrt{x}$ is continuous on $[0,2]$.
Solution: $\lim_{x\to a} \sqrt{x} = \sqrt{a}$ for every $a > 0$, it follows that f is continuous at a. Hence f is continuous on $(0,2)$.
Moreover, $\lim_{x\to 2^-} \sqrt{x} = \sqrt{2}$ and $f(2) = \sqrt{2}$.
Thus f is continuous from the left at 2.
Similarly,

$$\lim_{x\to 0^+} \sqrt{x} = 0 = f(0)$$

and so f is continuous from the right at 0. Hence f is continuous on $[0,2]$. ■

■ **Example 3.46** Let $f(x) = \frac{x^2-3x+2}{x^2+5x+6}$. Determine the numbers at which f is continuous.
Solution: Observe that f is a rational function. The denominator is 0 for $x = 1$ and $x = -6$, so f is defined for all x except 1 and -6. Therefore f is continuous at every number except 1 and -6. ■

Theorem 3.5.1 A function is continuous at **a** if and only if it is both continuous from the right and continuous from the left at **a**.

■ **Example 3.47** Determine the value of a constant a such that the function

$$f(x) = \begin{cases} ax^2+2 & \text{if } x \geq 3 \\ 2ax+11 & \text{if } x < 3 \end{cases}$$

is continuous at $x = 3$.
Solution: If $f(x)$ is to be continuous at $x = 3$, then $\lim_{x\to 3} f(x)$ must exist and furthermore

$$\lim_{x\to 3} f(x) = f(3)$$

$$\begin{aligned} \lim_{x\to 3^+} f(x) &= \lim_{x\to 3^+} (ax^2+2) = 9a+2 \text{ and} \\ \lim_{x\to 3^-} f(x) &= \lim_{x\to 3^+} (2ax+11) = 6a+11 \end{aligned}$$

Now equate the left and right side limits. This means

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x) = \lim_{x\to 3} f(x) = f(3)$$

From this it follows that $9a+2 = 6a+11$. Solving for a we get $a = 3$. Therefore the value of a must be 3 if $f(x)$ is to be continuous at $x = 3$. ■

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Theorem 3.5.2 If f and g are continuous at **a** and c is a constant, then $f \pm g$, cf, $f \cdot g$ and $\frac{f}{g}$, if $g(a) \neq 0$ are also continuous at **a**.

Theorem 3.5.3 If $f(x)$ is continuous at $x = \mathbf{a}$, then $|f(x)|$ is also continuous at $x = \mathbf{a}$.

Example 3.48 $f(x) = x$ is continuous at $x = 1$, then $|x|$ is also continuous at $x = 1$. ■

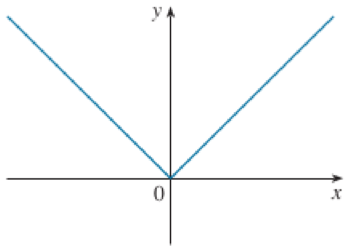

Figure 3.13: $f(x) = |x|$

Theorem 3.5.4 Polynomial functions, Rational functions, Root functions, Trigonometric functions, Inverse functions, Exponential functions and Logarithmic functions are continuous on their domain.

Theorem 3.5.5 If $\lim_{x\to\mathbf{a}} g(x) = b$ and f is continuous at b, then $\lim_{x\to\mathbf{a}} f(g(x)) = f(b)$; that is,

$$\lim_{x\to\mathbf{a}} f(g(x)) = f(\lim_{x\to\mathbf{a}} g(x))$$

Theorem 3.5.6 If g is continuous at **a** and f is continuous at $g(\mathbf{a})$, then fog is continuous at **a**; that is,

$$\lim_{x\to\mathbf{a}} f(g(x)) = f(g(\mathbf{a}))$$

Example 3.49 Show that h is continuous at 2, where $h(x) = \sqrt{x-1}$.
Solution: Let $g(x) = x - 1$ and $f(y) = \sqrt{y}$. Then $h = fog$. We know that g is continuous at $x = 2$ and that f is continuous at $g(2) = 1$. Since the square root function is continuous at every positive number. It follows from the above theorem that it is continuous at 2. ■

3.5.2 Intermediate Value Theorem

Theorem 3.5.7 Suppose f is continuous on a closed interval $[a,b]$. Let K be any number between $f(a)$ and $f(b)$, so that $f(a) \leq k \leq f(b)$ or $f(b) \leq k \leq f(a)$. Then there exist a number c in $[a,b]$ such that $f(c) = K$.

Notice that f be continuous on an interval I. If f has both positive and negative values on I, then the intermediate value theorem implies that $f(x) = 0$ for some x in I; that is, f has zero in I. Equivalently, if f has no zero in I, then either $f(x) > 0$ for all x in I or $f(x) < 0$ for all x in I.

Theorem 3.5.8 Let f be continuous on $[a,b]$ and $f(a) < 0 < f(b)$ or $f(b) < 0 < f(a)$, then the function have at least one solution.

Example 3.50 Show that the equation $x^5 - 2x^4 - 2x^3 + 8x^2 - 3x - 3$ has solution between 0 and 2.
Solution: Let

$$f(x) = x^5 - 2x^4 - 2x^3 + 8x^2 - 3x - 3$$

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since f is a polynomial function, then f is continuous on R. Here $f(0) = -3$ and $f(2) = 7$. Thus, $f(0) < 0 < f(2)$; that is, 0 is between $f(0)$ and $f(2)$. So, by Intermediate Value Theorem there is a number c between 0 and 2 such that $f(c) = 0$. ∎

∎ **Example 3.51** Let $f(x) = x^3 - x$. Find the solution of f on $[-2,2]$.
Solution: Here, $f(-2) = -6$ and $f(2) = 6$, because of this $f(-2) < 0 < f(2)$. By intermediate value theorem, there exist c in $[-2,2]$ such that $f(c) = 0$. So, c is the solution of f.

$$\begin{aligned} f(c) = c^3 - c = 0 &\Leftrightarrow c(c^2-1) = 0 \\ &\Leftrightarrow c(c-1)(c+1) = 0 \\ &\Leftrightarrow c = 0, c = 1 \text{ or } c = -1 \end{aligned}$$

Thus, -1, 0 and 1 are the solution of $f(x) = x^3 - x$. ∎

∎ **Example 3.52** If $f(x) = x^2 - x^2 + x$, show that there is a number c such that $f(c) = 10$.
Solution: The function $f(x)$ is continuous every where, so we can use the intermediate value theorem. Let take the interval $[0,3] \in R$, then at $x = 0$, $f(0) = (0)^3 - (0)^2 + 0 = 0$ and $x = 3$, $f(3) = (3)^3 - (3)^2 + 3 = 21$. Since $0 \leq f(c) \leq 21 \Rightarrow 0 \leq 10 \leq 21$, by Intermediate value theorem, there exist $0 \leq c \leq 21$, such that $f(c) = 10$. ∎

Theorem 3.5.9 Every continuous function in a closed interval $[a,b]$ attains its maximum and minimum value.

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3.6 Exercise

1. For the function g whose graph is given, state the value of each quantity, if it exist. If it does nor exist, explain why.

$(a)\lim_{x\to 0^-} g(x)$ $(b)\lim_{x\to 0^+} g(x)$ $(c)\lim_{x\to 0} g(x)$
$(d)\lim_{x\to 2^-} g(x)$ $(e)\lim_{x\to 2^+} g(x)$ $(f)\lim_{x\to 2} g(x)$
$(g)g(2)$ $(h)\lim_{x\to 4} g(x)$

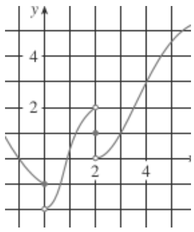

Figure 3.14: graphs of $g(x)$

Ans. $-1, -2, \text{does not exist}, 2, 0, \text{does not exist}, 1, 3$

2. Evaluate the infinite limit $\lim_{x\to 5^+} \frac{6}{x-5}$.

Ans. ∞

3. Evaluate the infinite limit $\lim_{x\to 5^+} \frac{x-1}{x^2(x+2)}$.

Ans. $-\infty$

4. Given that $\lim_{x\to 2} f(x) = 4$, $\lim_{x\to 2} g(x) = -2$, $\lim_{x\to 2} h(x) = 0$
Find the limits that exist. If the limits does not exist, explain why.

$(a)\lim_{x\to 2}[f(x)+g(x)]$ $(b)\lim_{x\to 2}[g(x)]^3$ $(c)\lim_{x\to 2}\sqrt{f(x)}$
$(d)\lim_{x\to 2}\frac{3f(x)}{g(x)}$ $(e)\lim_{x\to 2^+}\frac{g(x)}{h(x)}$ $(f)\lim_{x\to 2}\frac{g(x)h(x)}{h(x)}$

Ans. $-6, -8, 2, -6, \text{does not exist}, 0$

5. Evaluate the limit $\lim_{x\to 1}(\frac{1+3x}{1+4x^2+3x^4})^{\frac{1}{3}}$.

Ans. $\frac{1}{8}$

6. Evaluate the limit $\lim_{x\to 1}(\frac{1+3x}{1+4x^2+3x^4})^{\frac{1}{3}}$.

Ans. $\frac{1}{8}$

7. Evaluate the limit $\lim_{x\to 2}\frac{x^2+x-6}{x-2}$.

Ans. 5

8. Evaluate the limit $\lim_{x\to -4}(\frac{\frac{1}{4}+\frac{1}{x}}{4+x})$.

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Ans. $-\frac{1}{16}$

9. Let $f(x) = \frac{x^2-1}{|x-1|}$.
   (a) Find $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$.
   (b) Does $\lim_{x\to 1} f(x)$ exist?

Ans. 2, −2, No; does not exist

10. If $\lim_{x\to 1} \frac{f(x)-8}{x-1} = 10$, find $\lim_{x\to 1} f(x)$.

Ans. 8

11. For the limit $\lim_{x\to 1}(4+x-3x^3) = 2$ illustrate the definition limit by finding values of δ that corresponding to $\varepsilon = 1$ and $\varepsilon = 0.1$.

Ans. 0.11 and 0.012(or smaller positive numbers)

12. Prove that $\lim_{x\to 2}(x^2-4x+5) = 1$ by using $\varepsilon--\delta$ definition of limit.
13. Show that $f(x) = (x+2x^3)^4$ is continuous at $a = -1$.
14. Show that $f(x) = 2\sqrt{3-x}$ is continuous on the interval $(2,\infty)$.
15. Explain why the function $f(x) = \frac{x^4+17}{6x^2+x-1}$ is continuous at every number in its domain. State the domain.

Ans. $\{x|x \neq -\frac{1}{2}, \frac{1}{3}\}$

16. Show that $f(x)$ is containing on $(-\infty,\infty)$, where $f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ \sqrt{x} & \text{if } x \geq 1 \end{cases}$
17. Show that $f(x)$ is containing on $(-\infty,\infty)$, where $f(x) = \begin{cases} \sin x & \text{if } x < \frac{\pi}{4} \\ \cos x & \text{if } x \geq \frac{\pi}{4} \end{cases}$
18. Show that the function
$$f(x) = \begin{cases} x^4 \sin(\frac{1}{4}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$
is continuous on $(-\infty,\infty)$.
19. Use the intermediate value theorem to show that there is a root of the equation $x^4+x-3 = 0$ in the interval $(1,2)$.
20. Use the intermediate value theorem to show that there is a root of the equation $\sqrt[3]{x} = 1-x$ in the interval $(0,1)$.

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